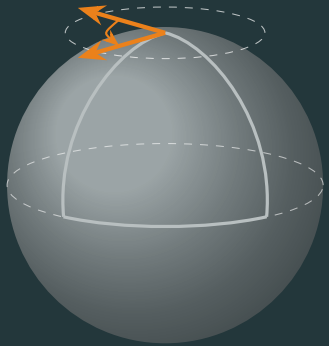
KCL/UCL JUNIOR GEOMETRY SEMINAR

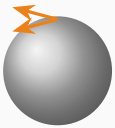
Special holonomy



Declan McGrath

King's College London · 27 October 2026

The plan



$d\omega = 0$ $\quad$ $\mathrm{Ric}(g) = 0$





I. What is it?

connections, holonomy, Berger's list

II. Kähler

Ricci form, $c_1(M)$

III. Does it exist?

Yau's theorem, Calabi–Yau metrics

IV. Can we see one?

explicit models, gluing

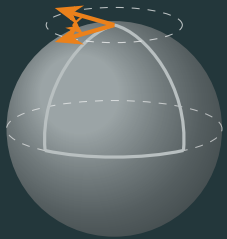
V. G_2

octonions, calibrations

I

What is it, and why care?

Connections, curvature, holonomy, Berger.



Why do we care?

Which manifolds carry a canonical metric, say Ricci-flat: $\mathrm{Ric}(g) = 0$?

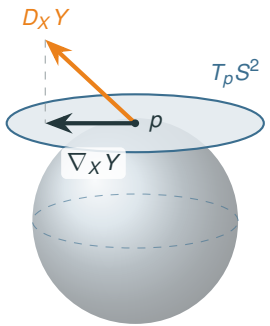
Algebraic geometers	**Complex geometers**	**Differential geometers**
Compact Kähler, K_X trivial $\Rightarrow$ Ricci-flat metrics (Yau)	Kähler geometry *is* a holonomy condition	Every known compact Ricci-flat manifold has special holonomy.

Special Holonomy appears to be the right setting for finding these canonical metrics

Differentiating on a manifold

- In $\mathbb{R}^n$ a vector field is n functions: differentiate each one to get $D_X Y$.
- On a manifold, T_pM and T_qM are different vector spaces: there is no canonical way to compare $Y(p)$ with $Y(q)$.
- On $S^2 \subset \mathbb{R}^3$ there is a natural fix: differentiate in $\mathbb{R}^3$ and project back onto the tangent plane,

$$\nabla_X Y := \left(D_X Y\right)^\top.$$



differentiate in $\mathbb{R}^3$ and keep the tangential part

Connections on vector bundles

Definition

A connection on a vector bundle $E \to M$ is an $\mathbb{R}$-linear map $\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibniz rule $\nabla(fs) = df \otimes s + f \nabla s$.

Why these axioms?

- These are the axioms the directional derivative D_X on $\mathbb{R}^n$ satisfies, and which the projection $(D_X Y)^\top$ on S^2 inherits.
- In a local frame they force $\nabla = d + A$, with A a matrix of 1-forms: a connection is a choice of A, and two connections differ by a tensor.

Levi-Civita

On TM with a metric there is a *unique* connection that preserves g and is torsion-free ($\nabla_X Y - \nabla_Y X = [X, Y]$). On S^2 it is the projection of the previous slide.

Parallel transport

Definition

A section s along a curve $\gamma : [0, 1] \to M$ is parallel if

$$\nabla_{\gamma'(t)} s = 0$$

In a local frame, with $\nabla = d + A$, this is the linear ODE

$$\dot{s}(t) + A\big(\gamma'(t)\big)\, s(t) = 0,$$

so given $v \in E_{\gamma(0)}$ there is a unique parallel section along γ with initial value v.

This defines the parallel transport map:

$$P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

Holonomy group

Fix a point $p \in M$ and consider loops $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p$.

Definition

The holonomy group at p is

$$\mathrm{Hol}_p(\nabla) = \{P_\gamma : E_p \to E_p \mid \gamma \text{ is a loop based at } p\}$$

This is a Lie subgroup of $\mathrm{GL}(E_p)$.

On a connected M, different basepoints give conjugate groups, so we often speak of the holonomy group $\mathrm{Hol}(\nabla)$.

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On a connected M, different basepoints give conjugate groups, so we often speak of the holonomy group $\mathrm{Hol}(\nabla)$.

Levi-Civita: transport preserves g, so $\mathrm{Hol}(\nabla) \subseteq O(n)$.

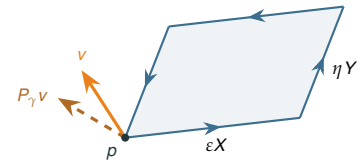
Curvature: transport round a small loop

Definition

The curvature of a connection ∇ is

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

For X, Y at p, $R(X, Y)$ is a linear map $E_p \to E_p$, antisymmetric in X, Y.



round the loop, v comes back slightly turned

Meaning: transport round a small parallelogram with sides εX, ηY is $P_\gamma = \mathrm{Id} - \varepsilon\eta\, R(X, Y) +$ (cubic terms in ε, η). Thus, one can think of curvature as capturing infinitesimal holonomy.

Examples (Levi-Civita): flat $\mathbb{R}^n$: $R = 0$. Unit sphere: $R(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$. On a surface, R is determined by the Gauss curvature K.

Ricci curvature

Definition

For the Levi-Civita connection, the Ricci tensor is a trace of R:

$$\mathrm{Ric}(X, Y) = \mathrm{tr}\left(Z \mapsto R(Z, X)Y\right), \qquad \mathrm{Ric}_{ij} = R_{kij}{}^{k}.$$

(M, g) is Ricci-flat if $\mathrm{Ric} \equiv 0$, and Einstein if $\mathrm{Ric} = \lambda g$.

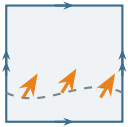
Meaning: Ricci curvature controls volume. In normal coordinates about p,

$$\sqrt{\det g} \;=\; 1 - \tfrac{1}{6}\,\mathrm{Ric}_{ij}(p)\, x^i x^j + O(|x|^3).$$

Examples: unit S^n: $\mathrm{Ric} = (n-1)g$. A surface: $\mathrm{Ric} = K\,g$. In dimensions 2 and 3, Ricci-flat $\Rightarrow$ flat; from dimension 4 on there are curved Ricci-flat metrics.

$\mathrm{Ric}(g) = 0$ *is the vacuum Einstein equation.*

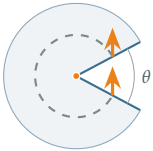
Holonomy: three examples



Flat torus

Round any loop, the vector returns unchanged.

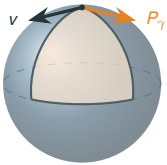
$\mathrm{Hol} = \{1\}$



Cone

Glue the two edges. Round the tip, the vector comes back turned by θ.

flat, yet Hol $\ni$ rotation by θ



Round sphere

Round the octant (three right angles), v turns by $\pi/2 =$ the area enclosed.

$\mathrm{Hol} = SO(2)$

Holonomy measures the curvature a loop encloses: on a surface, the angle is $\int K\, dA$.

Holonomy is generated by curvature

Theorem (Ambrose–Singer 1953)

The Lie algebra of $\mathrm{Hol}_p(\nabla)$ is spanned by the curvature operators $R_q(X, Y)$ at all points q, transported back to p along paths.[1]

Consequences:

- Flat $\iff$ the restricted holonomy $\mathrm{Hol}^0(\nabla)$, from null-homotopic loops, is trivial. (On the cone, remove the singular tip: the loop round it is not null-homotopic, so it can carry holonomy.)
- Constraints on curvature $\iff$ constraints on holonomy.

So which holonomy groups can actually occur?

[1] Ambrose, W. and Singer, I.M., *Trans. Amer. Math. Soc.* **75**(3), 428–443 (1953)

Special holonomy

For an n-dimensional orientable Riemannian manifold, generically $\mathrm{Hol}(\nabla) = SO(n)$.

Definition

A manifold has special holonomy if

$$\mathrm{Hol}(\nabla) \subsetneq SO(n),$$

i.e. a proper subgroup of $SO(n)$.

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Holonomy principle: *parallel tensor fields T ($\nabla T = 0$) on M $\longleftrightarrow$ tensors T_p on T_pM fixed by* $\mathrm{Hol}_p(\nabla)$. A smaller holonomy group fixes more tensors, and each extends by parallel transport to a parallel field.

Example: an orthogonal complex structure J_p ($J_p^2 = -1$) on $T_pM \cong \mathbb{R}^{2m}$ has stabiliser $U(m) \subset SO(2m)$. So

$$\mathrm{Hol} \subseteq U(m) \iff M \text{ carries a parallel } J.$$

Berger's classification

Theorem (Berger 1955)

For simply-connected Riemannian manifolds that are irreducible (not locally a product) and non-symmetric (not locally a symmetric space: $\nabla R \not\equiv 0$), the holonomy group is one of the following:

Holonomy	$\dim M$	**Type**	**Parallel structure**
$SO(n)$	n	generic	the metric only
$U(m)$	$2m$	Kähler	a complex structure
$SU(m)$	$2m$	Calabi–Yau*	$+$ a complex volume form
$Sp(m)$	$4m$	hyperkähler*	three complex structures
$Sp(m)\,Sp(1)$, $m \geq 2$	$4m$	quaternion-Kähler	a 4-form
G_2	7	$G_2{}^\star$	a 3-form
$\mathrm{Spin}(7)$	8	$\mathrm{Spin}(7)^\star$	a 4-form

* automatically Ricci-flat

II

Complex and Kähler geometry

$$J^2 = -\mathrm{Id}$$

$$d\omega = 0$$

Holonomy $U(m)$, and the condition $c_1 = 0$.

Complex manifolds

Definition

A complex manifold of complex dimension m is a manifold with an atlas of charts to $\mathbb{C}^m$ whose transition maps are holomorphic.

Multiplication by i in each chart gives a well-defined $J : TM \to TM$ with $J^2 = -\mathrm{Id}$.

Types: $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$, the $\pm i$-eigenspaces of J, spanned by the $\partial/\partial z^j$ and the $\partial/\partial \bar{z}^j$.
Complex forms split into (p, q)-forms, spanned by $dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{k_1} \wedge \cdots \wedge d\bar{z}^{k_q}$, and $d = \partial + \bar{\partial}$.

Hermitian metric: a Riemannian metric g with $g(JX, JY) = g(X, Y)$.

Almost complex structures

Definition

An almost complex structure is an endomorphism $J : TM \to TM$ with $J^2 = -\mathrm{Id}$. Its Nijenhuis tensor is

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].$$

Theorem (Newlander–Nirenberg 1957)

J comes from a complex structure if and only if $N_J \equiv 0$.[2]

Example: on an oriented surface with a metric, $J =$ rotation by 90°. In real dimension 2, $N_J \equiv 0$ automatically: every such J is a complex structure.

[2] Newlander, A. and Nirenberg, L., *Ann. of Math.* **65**(3), 391–404 (1957)

Kähler manifolds

The Kähler form of a Hermitian manifold: $\omega(X, Y) = g(JX, Y)$.

Definition

A Hermitian manifold (M, g, J) is called Kähler if $d\omega = 0$.

Equivalent conditions:

- $\nabla J = 0$ (the complex structure is parallel)
- $\mathrm{Hol} \subseteq U(m)$

Examples: $\mathbb{C}^m$; $\mathbb{C}P^m$ with the Fubini–Study metric; hence every smooth projective variety, by restriction.

The canonical bundle and holomorphic volume forms

Definition

The canonical bundle of a complex manifold M is

$$K_M = \Lambda^{m,0}\, T^*M,$$

the top exterior power of the holomorphic cotangent bundle.

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the top exterior power of the holomorphic cotangent bundle.

A holomorphic volume form is a nowhere-zero holomorphic section $\Omega \in \Gamma(K_M)$: it exists iff K_M is trivial.

Locally, $\Omega = f\, dz^1 \wedge \cdots \wedge dz^m$ with f holomorphic and non-vanishing.

Examples: on $\mathbb{C}^m$ and on tori $\mathbb{C}^m/\Lambda$, $\Omega = dz^1 \wedge \cdots \wedge dz^m$.

Adjunction: a smooth hypersurface $X \subset \mathbb{C}P^n$ of degree d has $K_X = \mathcal{O}(d-n-1)|_X$, so K_X is trivial when $d = n+1$: the quartic surface in $\mathbb{C}P^3$ (a K3), the quintic threefold in $\mathbb{C}P^4$.

The Ricci form is a curvature

On a Kähler manifold the Ricci form is $\rho(X, Y) = \mathrm{Ric}(JX, Y)$, a closed real $(1,1)$-form. In local coordinates, with $\omega = i\, g_{j\bar{k}}\, dz^j \wedge d\bar{z}^k$,

$$\rho = -i\, \partial\bar{\partial} \log \det(g_{j\bar{k}}).$$

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$$\rho = -i\,\partial\bar{\partial} \log \det(g_{j\bar{k}}).$$

Why this computes c_1:

1. Kähler $\Rightarrow$ Levi-Civita preserves $T^{1,0}M$ (as $\nabla J = 0$), so it induces a connection on the canonical bundle $K_M = \Lambda^{m,0} T^*M$. Since $|dz^1 \wedge \cdots \wedge dz^m|^2 = 1/\det(g_{j\bar{k}})$, its curvature is $F = \partial\bar{\partial} \log \det(g_{j\bar{k}}) = i\rho$.
2. Chern–Weil: for any connection on a line bundle L, the class $\left[\frac{i}{2\pi} F\right] \in H^2$ does not depend on the connection. It is the first Chern class $c_1(L)$; and $c_1(M) := c_1(T^{1,0}M) = -c_1(K_M)$.

$$[\rho] = -2\pi\, c_1(K_M) = 2\pi\, c_1(M) \in H^2(M, \mathbb{R}) \quad \text{for } \textit{every} \text{ Kähler metric on } M.$$

Example: Riemann surfaces

On a surface $\mathrm{Ric} = K\,g$, with K the Gauss curvature, so $\rho = K\,\omega$ and $[\rho] = 2\pi c_1$ says

$$\int_\Sigma K\,dA \;=\; 2\pi\,\chi(\Sigma) \;=\; 2\pi(2-2\gamma) \quad \text{for genus } \gamma \qquad \text{(Gauss–Bonnet)}.$$



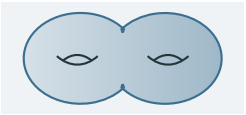
sphere, $c_1 > 0$

the round metric, $K > 0$



torus, $c_1 = 0$

a flat metric, $K = 0$



genus 2, $c_1 < 0$

a hyperbolic metric, $K < 0$

The sign of c_1 constrains which curvature is possible.

Holomorphic volume forms detect Ricci-flatness

If K_M is trivial, then $c_1 = 0$, so $[\rho] = 0$: ρ is *exact* for every Kähler metric. Can it be *zero*?

Holomorphic volume forms detect Ricci-flatness

If K_M is trivial, then $c_1 = 0$, so $[\rho] = 0$: ρ is *exact* for every Kähler metric. Can it be *zero*?

Write $\Omega = f\, dz^1 \wedge \cdots \wedge dz^m$. Then $|\Omega|_g^2 = |f|^2 / \det(g_{j\bar{k}})$ up to a constant, and $\partial\bar{\partial} \log |f|^2 = 0$, so

$$\rho \;=\; i\,\partial\bar{\partial} \log |\Omega|_g^2.$$

On compact M:

- $\rho \equiv 0 \iff |\Omega|_g$ is constant $\iff \nabla\Omega = 0 \iff \mathrm{Hol} \subseteq SU(m)$.
- So we want $\omega' \in [\omega]$ with $\omega'^m = c\,\Omega \wedge \bar{\Omega}$ for a constant c: prescribe the volume form.

Ricci-flat Kähler = the holomorphic volume form has constant length.

The $\partial\bar{\partial}$-lemma

Theorem ($\partial\bar{\partial}$-lemma)

On a compact Kähler manifold (M, ω), every d-exact (p, q)-form is also $\partial\bar{\partial}$-exact.

Consequence: two Kähler forms in the same class differ by $i\partial\bar{\partial}u$ for a single real function u.

III

Does it exist?

Yau's theorem and Calabi–Yau metrics.

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$$

$$\Downarrow \textit{ Yau}$$

$$\mathrm{Ric}(g) = 0$$

The Calabi conjecture

The Calabi conjecture (1954)

Let (M, J) be a compact complex manifold, and let g be a Kähler metric on M with Kähler form ω. Suppose ρ' is a real, closed $(1,1)$-form on M with $[\rho'] = 2\pi c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω' such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$ and the Ricci form of g' is ρ'.

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Remark (Calabi–Yau): if $c_1(M) = 0$, take $\rho' = 0$: a unique Ricci-flat Kähler metric in each Kähler class.

Yau's theorem

Theorem (Yau 1978)

The Calabi conjecture is true.

Writing $\omega' = \omega + i\partial\bar{\partial}u$ ($\partial\bar{\partial}$-lemma), the problem becomes a scalar PDE of complex Monge–Ampère type, with F a function fixed by ρ':

$$(\omega + i\partial\bar{\partial}u)^m = e^F \omega^m$$

- Calabi–Yau case: $e^F \omega^m = c\,\Omega \wedge \bar{\Omega}$, prescribing the volume form as on the Ricci-flatness slide.
- Proof: continuity method plus *a priori* estimates. It proves existence but gives no formula for the metric.

Calabi–Yau manifolds

Definition

A Calabi–Yau manifold is a compact Kähler manifold with K_M trivial (so $c_1(M) = 0$), carrying a Ricci-flat Kähler metric (Yau's theorem).

$$z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \subset \mathbb{C}P^3$$

the quartic K3: no formula known for its Ricci-flat metric

Key properties:

- Hol $\subseteq SU(m)$
- the holomorphic volume form Ω is parallel: $\nabla\Omega = 0$

Examples: complex tori, K3 surfaces (simply connected, K_M trivial, complex dimension 2), the quintic threefold in $\mathbb{C}P^4$ (adjunction: degree $n+1$ in $\mathbb{C}P^n$).

So what do these metrics actually look like?

IV

Can we see one?



Explicit models, and how to glue them in.

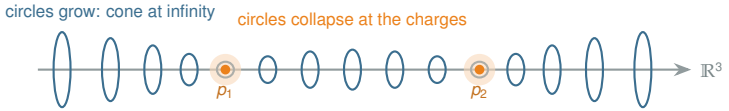
The Gibbons–Hawking recipe

To see one, give up compactness for now. In dimension four, the Calabi–Yau condition becomes a condition on three closed 2-forms ω_i: they must satisfy $\omega_i \wedge \omega_j = 2\delta_{ij}\,\mathrm{vol}$ (hyperkähler). A circle symmetry preserving them reduces this to Laplace's equation on $\mathbb{R}^3$.

Gibbons–Hawking 1978

$f > 0$ harmonic on $U \subseteq \mathbb{R}^3$ (a sum of point-charge potentials); a circle bundle with connection α, $d\alpha = -{*}df$. Then

$$g = f\,dx^2 + f^{-1}\alpha^2 \quad \text{has} \quad \mathrm{Hol}(g) \subseteq SU(2).$$



a circle bundle over $\mathbb{R}^3$; the circle over x has length $2\pi f(x)^{-1/2}$

What the recipe produces



one charge

$f = 1/(2|x|)$: flat $\mathbb{R}^4 = \mathbb{C}^2$; the circles form the Hopf fibration.



one charge + constant

$f = 1 + 1/(2|x|)$: Taub–NUT; the circles stay bounded at infinity.



$k+1$ charges

$f = \sum_i 1/(2|x - p_i|)$: the A_k ALE (asymptotically locally Euclidean) spaces, asymptotic to $\mathbb{C}^2/\mathbb{Z}_{k+1}$. Two charges: Eguchi–Hanson.

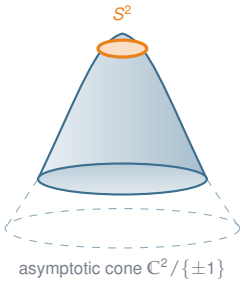
Two charges: Eguchi–Hanson

Eguchi–Hanson 1978

$$f = \frac{1}{2|x - p_1|} + \frac{1}{2|x - p_2|}$$

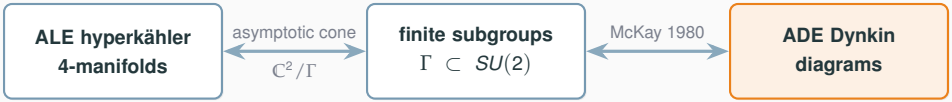
gives a complete metric on T^*S^2, asymptotic to the cone $\mathbb{C}^2/\{\pm 1\}$ with $O(r^{-4})$ decay.

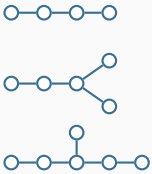
- the circles over the segment $[p_1, p_2]$, collapsing at its ends, sweep out a sphere



the cone point replaced by a sphere

Kronheimer: all of them





A_k: cyclic

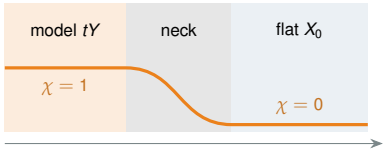
D_k: binary dihedral

E_6, E_7, E_8: binary tetra-, octa-, icosahedral

- A_k is Gibbons–Hawking with $k+1$ charges; D, E need Kronheimer's quotient construction (1989)
- a model for resolving every $\mathbb{C}^2/\Gamma$: one sphere per node, meeting as the diagram says

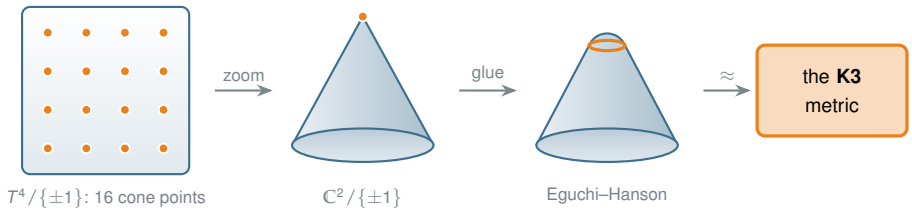
The gluing recipe

1. **Singular space** X_0: flat, with singularities modelled on $\mathbb{C}^2/\Gamma$.
2. **Model** Y: an ALE resolution of $\mathbb{C}^2/\Gamma$, shrunk by a factor t.
3. **Glue** with a cut-off in an annulus: an approximate solution g_t, with error $\to 0$ as $t \to 0$ because Y approaches the cone like r^{-4}.
4. **Perturb:** if the linearised equation is invertible with bounds uniform in t, the implicit function theorem gives an exact solution near g_t.



near each singular point, $g_t = \chi\, g_{tY} + (1-\chi)\, g_{\text{flat}}$

The recipe in action: Kummer's K3



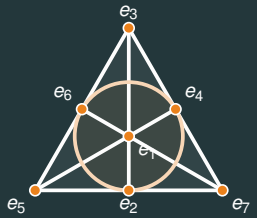
one Eguchi–Hanson at each cone point (Page 1978; Donaldson 2012)

Yau's theorem gives us that the K3 metric exists, the above models what this metric looks like.

V

G_2

The geometry of the octonions.



The four division algebras

Definition

A normed division algebra is $\mathbb{R}^n$ with a bilinear product, a unit, and $|xy| = |x|\,|y|$.

Theorem (Hurwitz 1898)

There are exactly four: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, of dimensions $1, 2, 4, 8$.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$SO(n)$	$U(m)$, $SU(m)$	$Sp(m)$, $Sp(m)Sp(1)$	G_2, $\mathrm{Spin}(7)$
generic metrics (Part I)	Kähler, Calabi–Yau (Parts II–III)	hyperkähler (Part IV)	the exceptional rows (now)

Berger's list lines up with the division algebras. We will now focus on those corresponding to the octonions.

Cross products exist only in dimensions 3 and 7

Definition

A cross product on $\mathbb{R}^n$: a bilinear map $u \times v$, orthogonal to u and v, with $|u \times v|^2 = |u|^2|v|^2 - \langle u, v\rangle^2$.

Theorem (Brown–Gray 1967; via Hurwitz)

Non-zero cross products exist only on $\mathbb{R}^3$ and $\mathbb{R}^7$.

$\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$: symmetries $SO(3)$

$\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$: symmetries G_2

$u \times v = \mathrm{Im}(uv)$;

$\dim G_2 = 14 < 21 = \dim SO(7)$

G_2 is to the seven-dimensional cross product what $SO(3)$ is to the familiar one.

Linear algebra on $\mathbb{R}^7$

Turn the cross product into a 3-form:

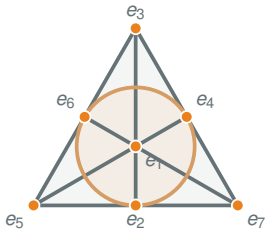
$$\varphi_0(u, v, w) := \langle u \times v, w \rangle$$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

$G_2 := \mathrm{Stab}(\varphi_0) = \mathrm{Aut}(\mathbb{O})$

compact, simple, $\dim G_2 = 14$. Count:
$49 - 14 = 35 = \dim \Lambda^3(\mathbb{R}^7)^*$, so G_2-type 3-forms form an open set.

Exactly as $\omega(u, v) = \langle Ju, v \rangle$ on $\mathbb{C}^m$:
same construction, one slot more.



the Fano plane: the multiplication table of $\mathrm{Im}\,\mathbb{O}$*;*
one term of φ_0 *per line*

The odd one out

ω

a 2-form on $\mathbb{R}^{2m}$

needs J: $g = \omega(\cdot, J\cdot)$

Kähler geometry: ω and J are separate data, forced to agree

φ

a 3-form on $\mathbb{R}^7$

no extra data needed:
$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 6\, g(u, v)\ \mathrm{vol}$

the 3-form alone determines the metric

- a G_2-structure on M^7: a 3-form φ that at each point looks like φ_0, i.e. a cross product on every tangent space
- torsion-free: $\nabla\varphi = 0$, equivalently $\mathrm{Hol} \subseteq G_2$; then g is Ricci-flat
- Fernández–Gray 1982: $\nabla\varphi = 0 \iff d\varphi = 0 = d{*}\varphi$; nonlinear, as $*$ depends on φ

Why G_2 geometry is interesting

Ricci-flat in odd dimensions

Torsion-free G_2 metrics are Ricci-flat. They are the only known source of Ricci-flat metrics on compact, simply connected, odd-dimensional manifolds.

No complex structure

Without J there is no $\partial\bar{\partial}$-lemma, so the equations do not reduce to one scalar PDE: no Yau theorem.

Physics

M-theory on $\mathbb{R}^4 \times (G_2 \text{ manifold})$ plays the role of string theory on $\mathbb{R}^4 \times (\text{Calabi–Yau 3-fold})$: supersymmetric physics in four dimensions.

Special submanifolds

φ and $*\varphi$ single out *associative* 3-folds and *coassociative* 4-folds, which minimise volume (end of the talk).

G_2 geometry is Calabi–Yau geometry without the complex structure.

G_2 contains the geometries we have seen

Split $\operatorname{Im}\mathbb{O} = \operatorname{Im}\mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^3 \oplus \mathbb{R}^4$. Then

$$\varphi_0 = dx^{123} + dx^1 \wedge \omega_1 + dx^2 \wedge \omega_2 + dx^3 \wedge \omega_3,$$

with $(\omega_1, \omega_2, \omega_3)$ the hyperkähler triple on $\mathbb{R}^4$.

Hyperkähler $\times$ flat

X^4 hyperkähler: $T^3 \times X$ carries the torsion-free

$$\varphi = dx^{123} + \textstyle\sum_i dx^i \wedge \omega_i,$$

with Hol $\subseteq Sp(1) \subset G_2$.

Calabi–Yau $\times$ circle

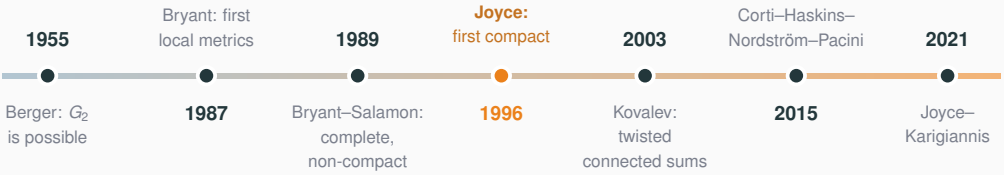
Y^6 Calabi–Yau: $S^1 \times Y$ carries the torsion-free

$$\varphi = d\theta \wedge \omega + \operatorname{Re}\Omega,$$

with Hol $\subseteq SU(3) \subset G_2$.

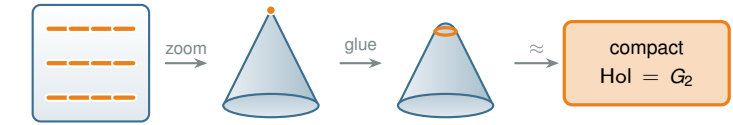
In particular $T^3 \times$ Eguchi–Hanson is a G_2 manifold: Part IV's models live here.

Do G_2 manifolds exist? A timeline



Every compact example known today is built by gluing, then perturbing.

Compact G_2 manifolds: Joyce's construction



Joyce 1996: resolve T^7/Γ with Part IV's ALE models: the first compact G_2 manifolds

- $\Gamma \cong (\mathbb{Z}/2)^3$ acts on T^7 preserving the flat φ_0; the quotient is singular along 12 disjoint 3-tori.
- Near each, the picture is $T^3 \times (\mathbb{C}^2/\{\pm 1\})$; replace it by $T^3 \times$ Eguchi–Hanson, a G_2 manifold by the product construction.
- Joyce's existence theorem: a closed G_2-structure with small enough torsion (in suitable norms) can be perturbed to a torsion-free one.

Kummer and Joyce: one recipe

	Kummer (dimension 4)	Joyce (dimension 7)
1. singular space	$T^4/\{\pm 1\}$, 16 points	T^7/Γ, 12 copies of T^3
local model	$\mathbb{C}^2/\{\pm 1\}$	$T^3 \times \mathbb{C}^2/\{\pm 1\}$
2. resolution	Eguchi–Hanson	$T^3 \times$ Eguchi–Hanson
3. glued structure	hyperkähler triple, small error	closed φ, small torsion
4. perturb	implicit function theorem	Joyce's existence theorem
result	K3, Hol $= SU(2)$	compact, Hol $= G_2$

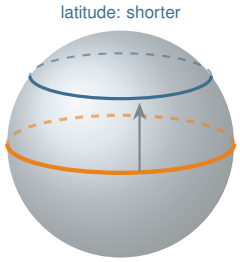
Part IV's models are exactly what dimension seven needs.

Inside the metrics: minimal submanifolds

Geodesics are critical points of length. Their higher-dimensional analogue: minimal submanifolds, critical points of volume, i.e. mean curvature $H = 0$.

Two problems:

1. $H = 0$ is a second-order nonlinear PDE system. Even for a graph $x_n = f(x_1, \ldots, x_{n-1})$ it reads $\operatorname{div}\left(\nabla f / \sqrt{1 + |\nabla f|^2}\right) = 0$.
2. Critical does not mean minimising: the equator of S^2 is a geodesic, but nearby circles of latitude are shorter.[3]



a critical point of length, but not a minimum

Calibrations are useful because they provide a first-order condition that certifies global minimality.

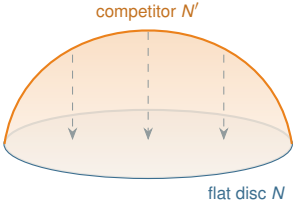
[3] J. D. Lotay, *Calibrated submanifolds*, arXiv:1810.08709.

What is a calibration?

Definition (Harvey–Lawson 1982)

A calibration is a closed p-form α with $\alpha|_\xi \le \mathrm{vol}_\xi$ for every oriented tangent p-plane ξ. A submanifold N is calibrated if $\alpha|_N = \mathrm{vol}_N$.

Simplest example: on $\mathbb{R}^n$, $\alpha = dx^1 \wedge \cdots \wedge dx^p$. For a p-plane ξ, $\alpha|_\xi$ is the area factor of projecting ξ onto $\mathbb{R}^p$: at most 1, with equality iff ξ is parallel to $\mathbb{R}^p$ and oriented the same way.



same boundary; projecting onto the plane shrinks area, so $\mathrm{vol}(N') \ge \mathrm{vol}(N)$

Calibrated submanifolds minimise volume

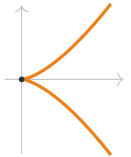
Theorem (Harvey–Lawson 1982)

If N is calibrated by α and N' is homologous to N (or has the same boundary), then, since $d\alpha = 0$,

$$\mathrm{vol}(N) = \int_N \alpha = \int_{N'} \alpha \leq \mathrm{vol}(N').$$

Example (Wirtinger): on a Kähler manifold, $\omega^k / k!$ calibrates exactly the complex submanifolds; in complex dimension one, the holomorphic curves.

So holomorphic curves in $\mathbb{C}^2$ minimise area for their boundary, even singular ones like the cusp $\{z^2 = w^3\}$.



real points of $z^2 = w^3$

Special holonomy gives calibrations

A parallel form is closed, and $\max_\xi \alpha|_\xi / \operatorname{vol}_\xi$ is the same at every point: rescaled, every parallel form is a calibration.

The holonomy group moves calibrated planes around, so there are many of them.

geometry	form	calibrates
Kähler	$\omega^k / k!$	complex
Calabi–Yau	$\operatorname{Re}\Omega$	special Lagrangian
G_2	$\varphi,\ *\varphi$	(co)associative
Spin(7)	Φ	Cayley

Special holonomy provides high-codimension minimal submanifolds.

Associatives: the G_2 calibration

Theorem (Harvey–Lawson 1982)

On any oriented 3-plane $\xi \subset \mathbb{R}^7$:

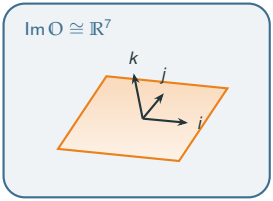
$$\varphi_0|_\xi \leq \mathrm{vol}_\xi,$$

with equality iff ξ is a copy of $\mathrm{Im}\,\mathbb{H} \subset \mathrm{Im}\,\mathbb{O}$: an associative plane.

Examples in $S^1 \times Y^6$, $\varphi = d\theta \wedge \omega + \mathrm{Re}\,\Omega$:

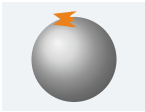
- $S^1 \times \Sigma$, Σ a holomorphic curve
- $\{\theta\} \times L$, L special Lagrangian

And: the zero section S^3 of the Bryant–Salamon spinor bundle (1989) is associative.



$\mathrm{Im}\,\mathbb{H} \subset \mathrm{Im}\,\mathbb{O}$: *the model associative plane*

What to take home



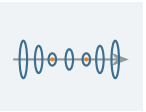
Holonomy

Transport around loops: a group. Smaller group = parallel structure.



Berger

The list can be organised by the four division algebras.



Existence

Yau: exists, no formula. Explicit models plus gluing: what it looks like.



G_2

A cross product on $\mathbb{R}^7$. One 3-form gives the metric and calibrates associatives.

Back to the opening question: for compact Kähler manifolds the answer is $c_1 = 0$ (Yau). For G_2, nobody knows: we can only build examples.

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Thank you.